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Generalized theory of multilayer plates

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Abstract

The layer-wise generalized theory of elastodynamic of multilayer plates is presented in this paper. This theory is based on expanding the displacement vector components of each layer into power series about the transverse coordinate. The number of terms retained in the power series is arbitrary and it is chosen depending on the problem being considered and the solution accuracy required. The system of governing equations is obtained by Hamilton's variation principle.

The possibilities of the theory proposed and validity of results obtained are illustrated by examples of investigating the strain-stressed state of one- and three-layer structures. The issues of applicability of two-dimensional approximations built on the basis of the power series method are considered with respect to calculation of displacements, inplane and transverse stresses in multilayer plates under dynamic loading. Calculation results are compared with data obtained from Ambartsumyan's theory (the hypothesis of a unique non-strained normal for the pack), the layer-wise theory based on the broken line hypothesis as well as the three-dimensional elasticity theory. © 2002 Published by Elsevier Science Ltd.

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1. Introduction

Over the past decades a dramatic development of computational tools and mathematical methods for analyzing the dynamics of multilayer structures has been observed (Whitney and Sun, 1977; Librescu and Reddy, 1986; Di Sciuva, 1986; Thangjitham et al., 1987; Reddy, 1993; Nosier et al., 1994; Cupial and Niziol, 1995; Smetankina et al., 1995; Tessler et al., 1995; Cheng et al., 1996; Shupikov et al., 1998; Shupikov and Smetankina, 2001). Progress in this field, on the one hand, has been brought about by the necessity of solving new application problems, and on the other hand it became possible because of the truly fantastic developments in computers.

The key feature of the modern stage of development of the mechanics of multilayer structures consists in the transition from more simple models to more complex ones possessing higher accuracy and universality. Apparently, this trend will be sustained in the near future and one of the basic lines of research in

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computational mechanics of multilayer structures will be the development of mathematical models and algorithms making it possible to more accurately describe the processes taking place during deformation in actual structures.

When considering an elastic material with sufficiently small deformations, the standard of accuracy is the classical equations of elasticity (Novatsky, 1975). However, the application of three-dimensional elasticity theory equations for investigating non-stationary deformation in multilayer structures is faced with significant mathematical complexities (Shupikov and Ugrimov, 1999). Therefore, two-dimensional theories have found extensive implementation in the design of real structures (see, for example, Reddy, 1993; Smetankina et al., 1995; Tessler et al., 1995; Shupikov et al., 1998; Shupikov and Smetankina, 2001).

As a rule, two-dimensional approximations are built for a specific class of structures, e.g. three-layer structures with rigid or soft fillers (Grigoliuk and Chulkov, 1964; Prusakov, 1951) and multilayer structures having a definite (symmetric, regular) pack structure (Lekhnitskii, 1941; Moskalenko and Novichkov, 1967). In doing so, the transition to another class of structures is not always admissible since this may lead to an inadequate description of the structure's behavior or, what is also important, to excessive computational input. Hence, it is necessary to develop universal theories having a high accuracy when describing a wide class of structures, algorithmic flexibility, simplicity and efficiency.

One of the basic methods of building two-dimensional approximations is the power series method. This method dates back to the works of the renowned mathematicians Cauchy (1828) and Poisson (1829) who suggested to expand stress vector components into power series about the transverse variable, which characterizes the position of an arbitrary point with respect to the middle surface. In doing so, the assumption on convergence of these series was introduced. Later on, this approach was extended by Krauss (1929) and Kilchevsky (1939) to the general theory of shell statics. This approach was developed by Epstein (1942) in the dynamics of plates and shells. At present, theories based on the assumed stress field are used less frequently in practice because they are more complex in their numerical implementation as compared to the displacement-based theories (Reddy, 1993). Further, the work discusses the genesis of displacement-based theories based on the power series method and their application ranging from homogeneous to multilayer structures.

The power series method has enjoyed wide application for building applied displacement-based mathematical theories of homogeneous plates and shells. Thus, in the well-known Mindlin plate theory based on Timoshenko type hypotheses it is assumed that the plate's displacement vector components have the following form (Mindlin, 1951)

$$u = u_0 + z\psi_x, \quad v = v_0 + z\psi_y, \quad w = w_0, \tag{1}$$

where z is the transverse coordinate; u_0, v_0, w_0 are displacements of a point on reference surface in the direction of the coordinate axes; ψ_x, ψ_y are the angles of rotation of a transverse normal about axes Ox and Oy, correspondingly. The functions u_0, v_0, w_0, ψ_x, ψ_y are the sought for functions depending on plane coordinate x, y. This approach was extended by Mirsky and Herrmann (1957) to the theory of cylindrical shells.

Hypotheses (1) may be considered as power series expansions of displacements about the transverse coordinate. In doing so, two expansion terms are retained for the inplane displacements and only one term is retained for the transverse displacement. According to the usual tradition (for example, Reddy, 1989; Tessler et al., 1995), in the following we will designate models by the maximum power of the transverse coordinate retained in the series for tangential and transverse displacements. Thus, the hypotheses of Mindlin plate theory being considered will be designated as $\{1, 0\}$.

Taking into account the transverse shear strain in hypothesis (1), provides this theory with a wider range of applicability as compared to Kirchhoff classical theory. However, the incorrectness of the approximation $w = w_0$ in hypotheses (1), which was the same as in Kirchhoff classical theory, as well as no allowance for the curved normal in the model, has forced the investigators to develop advanced refined theories.

A number of refined models were developed, which take into account a greater number of terms in the power series and which have wider applicability limits.

Naghdi (1957) derived the equations of the general theory of shells, which take into account the influence of transverse normal and shear strains. The displacements were approximated by the following expressions:

$$u = u_0 + z\psi_x, \quad v = v_0 + z\psi_y, \quad w = w_0 + z\psi_z + z^2\xi_z.$$

Lo et al. (1977) developed the theory {3, 2} which describes the spatial character of strain with greater accuracy (non-linear distribution of the displacements and parabolic distribution of the transverse shear stresses with respect to the thickness coordinate). The authors proceeded from the approximation of displacement vector components with finite power series. In doing so, the inplane components changed according to the cubic law and the transverse ones changed according to the quadratic law:

$$\begin{aligned} u &= u_0 + z\psi_x + z^2\xi_x + z^3\Phi_x, \\ v &= v_0 + z\psi_y + z^2\xi_y + z^3\Phi_y \\ w &= w_0 + z\psi_z + z^2\xi_z. \end{aligned} \tag{2}$$

The theory allows for the affect of the shear strains and the transverse normal strains, as well as of the normal curvature. This is especially significant when considering structures having a relatively big thickness as well as during localized loading.

Analyzing the characteristics of the models being considered, one may assert the following: an increase in the number of terms retained in the power series leads to a more accurate description of the plate strain process. The more detailed review of refined models for homogeneous construction can be found, for example, in the work of Lo et al. (1977).

The power series method is used widely in the theory of multilayer plates and shells as well.

The multilayer structure has a considerable effect on the requirements to the theory being used. The presence of layers in the pack, which have significantly differing physical characteristics, makes the structure all the more susceptible to transverse strain, viz. shear and reduction. The deflected mode of multilayer structures has a spatial character defined by the curvature and length variations of the normals to the external surfaces. A no less important factor of a well-posed statement of the problem of multilayer structure strain is the strict implementation of boundary conditions, as well as the interlayer contact conditions. Unfortunately, in spite of extensive investigation in this area, the dynamics of multilayer structures has not yet been investigated sufficiently.

In the theory of multilayer structures there exist two approaches to building two-dimensional theories of multilayer plates and shells (Grigoliuk and Kogan, 1972; Grigoliuk and Kulikov, 1988; Reddy, 1989; Reddy, 1993; Grigoliuk and Kogan, 1998), viz.:

- the first, more general approach, uses hypotheses for each separate layer to derive equations (Reddy, 1989; Reddy, 1993; Smetankina et al., 1995; Shupikov et al., 1998);
- the second approach uses hypotheses for the pack as a whole to derive the governing equations (Ambartsumyan, 1974; Reddy, 1989; Reddy, 1993; Tessler et al., 1995).

In the theory of composite plate and shells these approaches are called layer-wise and equivalent single-layer respectively. A review of the layer-wise and equivalent single-layer theories contains in works of Noor and Burton (1989), Reddy (1990) and Grigoliuk and Kogan (1999).

Each approach has its advantages and deficiencies. Thus, the number of governing equations in the first approach depends on the number of layers, which increases the problem complexity. This approach, however, makes it possible to obtain a more accurate description of strains and stresses in each pack layer, as well as of the interlayer contact conditions.

The number of the equations in the second approach does not depend on number of layers. However, in all single-layer displacement-based theories the displacements and strains are continuous through the thickness of pack. This leads to discontinuous stress field through the thickness because different mechanical characteristics of layers are used to compute stresses. Thus, within the framework of single-layer displacement-based theories it is impossible to describe accurately the interlayer contact stresses as well as the transverse strains over the pack thickness.

The power series method has found its application in both approaches.

Tessler et al. (1995), in investigating the vibrations of thick multilayer plates, used the equivalent single-layer theory based on the model {1, 2}. Lo et al. (1977) developed the equivalent single-layer third-order theory based on hypothesis (2) for investigating the behavior of composites. Reddy's third-order theory is based on approximation of the displacement field by the expressions

$$u_1(x,y,z) = u + z\phi_1 - z^3\frac{4}{3h^2}\left(\phi_1 + \frac{\partial w}{\partial x}\right), \quad u_2(x,y,z) = v + z\phi_2 - z^3\frac{4}{3h^2}\left(\phi_2 + \frac{\partial w}{\partial y}\right), \quad u_3(x,y,z) = w,$$

which may be considered as power expansions of the displacement vector components.

The best-known *layer-wise theory* of a multilayer structure is based on the broken line hypothesis (Cupial and Niziol, 1995; Smetankina et al., 1995). One of the variants of this theory is Grikoliuk's model (Grigoliuk and Chulkov, 1964; Smetankina et al., 1995), in which the displacements of each plate layer are described by hypothesis (1):

$$\begin{aligned}
u^i(x,y,z,t) &= u_0 + \sum_{j=1}^{i-1} h_j\psi_{xj} + (z-\delta_{i-1})\psi_{xj},\\
v^i(x,y,z,t) &= v_0 + \sum_{j=1}^{i-1} h_j\psi_{yj} + (z-\delta_{i-1})\psi_{yj},\\
w^i(x,y,z,t) &= w_0.
\end{aligned} \tag{3}$$

Here u^i, v^i, w^i are the displacements of a point in the ith layer in the directions of the coordinate axes, t is the time, h_i is the thickness of the ith layer, $\delta_i = \sum_{j=1}^{i} h_j$.

The deficiency of this model is the transverse displacements approximation used, which leads to an absence of pack reduction over the thickness. This model turns out to be inefficient under localized effects on the plate characterized by a non-linear dependence of displacements and stresses on the transverse coordinate (Shupikov et al., 1998), as well as the presence of soft fillers in the pack. Shupikov et al. (1998) suggested a model which eliminates the deficiencies of this theory. The model describes the behavior of each layer by using hypothesis (2). This theory takes into account transverse shear and normal strains and also normal curvature in each layer.

Analysis of the characteristics of the multilayer structure models considered has shown that, similar to the case of homogeneous structures, an increase in the number of retained terms leads to an increase in the model's accuracy, which ultimately leads to an extension of model application limits.

This work proposes a generalized displacement-based theory of multilayer plates based on kinematical hypotheses for each separate layer. The kinematical hypotheses are the expansions of the displacement vector components for each layer into finite power series about the transverse coordinate.

2. Problem statement

A multilayer plate consists of I layers of constant thickness, h_i is the thickness of the ith layer. Each layer of plate is made from a homogeneous isotropic material. The mechanical parameters of the ith layer of

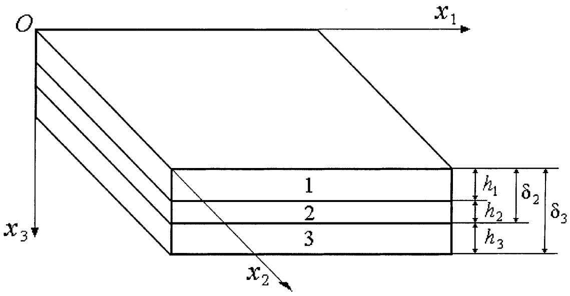

Fig. 1. Multilayer plate.

plate we shall designate as follows: E_i is Young's modulus, ν_i is Poisson's ratio, ρ_i is the density. It is assumed that contact between layers excludes their delamination and mutual slipping.

The plate is referred to the Cartesian coordinate system $Ox_1x_2x_3$ and the coordinate plane Ox_1x_2 is connected with the outside surface of the first layer (Fig. 1).

The external force $\bar{q} = \bar{q}(x_1, x_2, t)$ is applied on outside surface of the first layer; q_α, $\alpha = 1, 2, 3$ are the coordinate axes projections of this force.

The behavior of a multilayer plate is described by the equations of the generalized theory of multilayer plates. This theory is based on the assumption that the displacements of each plate layer may be presented in the form of finite power series about the transverse coordinate x_3.

The displacements of a point ith layer are described by the following kinematic relationships:

$$\begin{aligned}
u_\nu^i(x_1, x_2, x_3, t) &= u_\nu + \sum_{k=1}^{K}\left[\sum_{j=1}^{i-1} h_j^k u_{\nu k}^j + (x_3 - \delta_{i-1})^k u_{\nu k}^i\right], \quad \nu = 1, 2, \\
u_3^i(x_1, x_2, x_3, t) &= u_3 + \sum_{\ell=1}^{L}\left[\sum_{j=1}^{i-1} h_j^\ell u_{3\ell}^j + (x_3 - \delta_{i-1})^\ell u_{3\ell}^i\right],
\end{aligned} \tag{4}$$

where

$$h_j^k = (h_j)^k, \quad \delta_i = \sum_{j=1}^{i} h_j, \quad \delta_{i-1} \leqslant x_3 \leqslant \delta_i, \qquad i = \overline{1, I},$$

$u_\alpha^i(\alpha = 1, 2, 3)$ are displacements of a point ith layer in the direction of the coordinate axes Ox_α; u_ν, u_3, $u_{\nu k}^i$, $u_{3\ell}^i$ are the terms in a power series expansions in x_3 depending on x_1, x_2, t; K, L are maximum powers of terms retained in a power series for inplane and transverse displacements, accordingly. The values of parameters K, L are chosen depending on the approximation accuracy required. Thus, in the generalized theory of multilayer plates the behavior of each layer is described by the theory $\{K, L\}$.

The kinematic relationships (4) are equivalent to hypothesis (3) (broken line hypotheses) and hypotheses of the theory proposed by Shupikov et al., 1998 at $K = 1$, $L = 0$ and $K = 3$, $L = 2$, correspondingly.

The strains of the layers are assumed small and are described by Cauchy's formulas:

$$\varepsilon_{\alpha\beta}^i = \frac{1}{2}\left(u_{\alpha,\beta}^i + u_{\beta,\alpha}^i\right), \quad \alpha = 1, 2, 3, \ \ \beta = 1, 2, 3, \ \ i = \overline{1, I}.$$

The stresses in the layers are calculated on the basis of the Hooke's law

$$p^i_{\alpha\beta} = \lambda^i \delta_{\alpha\beta}\varepsilon^i_{ll} + 2\mu^i \varepsilon^i_{\alpha\beta}, \qquad \varepsilon^i_{ll} = \varepsilon^i_{11} + \varepsilon^i_{22} + \varepsilon^i_{33}, \quad \alpha = 1,2,3, \;\; \beta = 1,2,3, \;\; i = \overline{1,I},$$

where $\delta_{\alpha\beta}$ is Kronecker delta, $\lambda^i = E_i v_i/(1+v_i)(1-2v_i)$, $\mu^i = E_i/2(1+v_i)$.

Note that the application of hypothesis (4) yields a continuous displacement field over the pack thickness and ensures a continuity of inplane strain over the pack thickness and piecewise-continuity of the transverse strains. Thus, within the framework of the generalized theory there exists the principal possibility of fulfilling the requirements of the interlayer contact conditions accurately.

The stress resultants in the ith layer are determined under the following formulas

$$N^{ik}_{\alpha\beta} = N^{ik}_{\beta\alpha} = \int_{\delta_{i-1}}^{\delta_i} (x_3 - \delta_{i-1})^k p^i_{\alpha\beta}\,\mathrm{d}x_3, \quad \alpha,\beta = 1,2,3, \;\; i = \overline{1,I}.$$

3. Equations of motion, boundary and initial conditions

The governing equations have been derived from a Hamilton's variational principle. The equations of motions in term of stress resultant have the following form

$$\begin{aligned}
&\sum_{i=1}^{I}\left[L^i_1 - I^i_{11}\right] + q_1 = 0, \quad \sum_{i=1}^{I}\left[L^i_2 - I^i_{21}\right] + q_2 = 0, \quad \sum_{i=1}^{I}\left[L^i_3 - I^i_{31}\right] + q_3 = 0,\\
&N^{ik}_{11,1} + N^{ik}_{12,2} - kN^{ik-1}_{13} + h^k_i \sum_{j=i}^{I-1}\left[L^{j+1}_1 - I^{j+1}_{11}\right] - I^i_{1k+1} = 0,\\
&N^{ik}_{12,1} + N^{ik}_{22,2} - kN^{ik-1}_{23} + h^k_i \sum_{j=i}^{I-1}\left[L^{j+1}_2 - I^{j+1}_{21}\right] - I^i_{2k+1} = 0,\\
&N^{i\ell}_{13,1} + N^{i\ell}_{23,2} - \ell N^{i\ell-1}_{33} + h^\ell_i \sum_{j=i}^{I-1}\left[L^{j+1}_3 - I^{j+1}_{31}\right] - I^i_{3\ell+1} = 0, \qquad k = \overline{1,K}, \;\; \ell = \overline{1,L}, \;\; i = \overline{1,I},
\end{aligned} \tag{5}$$

where

$$L^i_1 = N^{i0}_{11,1} + N^{i0}_{12,2}, \quad L^i_2 = N^{i0}_{22,2} + N^{i0}_{12,1}, \quad L^i_3 = N^{i0}_{13,1} + N^{i0}_{23,2},$$

$$I^i_{vr} = \frac{\rho_i h^r_i}{r}\left(u_{v0,tt} + \sum_{k=1}^{K}\left[\sum_{j=1}^{i-1} h^k_j u^j_{vk,tt} + \frac{rh^k_i}{k+r} u^i_{vk,tt}\right]\right), \quad v = 1,2,$$

$$I^i_{3r} = \frac{\rho_i h^r_i}{r}\left(u_{30,tt} + \sum_{\ell=1}^{L}\left[\sum_{j=1}^{i-1} h^\ell_j u^j_{3\ell,tt} + \frac{rh^\ell_i}{\ell+r} u^i_{3\ell,tt}\right]\right).$$

Thus, dynamic response of a plate is described by $(2K+L)I+3$ differential equations (5).

Eq. (5) are associated with the following boundary condition, which were also obtained from the variational principle

$$\begin{aligned}
&\sum_{i=1}^{I} N_{nn}^{i0} = \sum_{i=1}^{I} N_{nn}^{i0^*} \ \text{ or } \ u_n = u_n^*, \\
&\sum_{i=1}^{I} N_{n\tau}^{i0} = \sum_{i=1}^{I} N_{n\tau}^{i0^*} \ \text{ or } \ u_\tau = u_\tau^*, \\
&\sum_{i=1}^{I} N_{n3}^{i0} = \sum_{i=1}^{I} N_{n3}^{i0^*} \ \text{ or } \ u_3 = u_3^*, \\
&N_{nn}^{ik} + h_i^k \sum_{j=i}^{I-1} N_{nn}^{j+1\,0} = N_{nn}^{ik^*} + h_i^k \sum_{j=i}^{I-1} N_{nn}^{j+1\,0^*} \ \text{ or } \ u_{nk}^i = u_{nk}^{i^*}, \\
&N_{n\tau}^{ik} + h_i^k \sum_{j=i}^{I-1} N_{n\tau}^{j+1\,0} = N_{n\tau}^{ik^*} + h_i^k \sum_{j=i}^{I-1} N_{n\tau}^{j+1\,0^*} \ \text{ or } \ u_{\tau k}^i = u_{\tau k}^{i^*}, \\
&N_{n3}^{i\ell} + h_i^\ell \sum_{j=i}^{I-1} N_{n3}^{j+1\,0} = N_{n3}^{i\ell^*} + h_i^\ell \sum_{j=i}^{I-1} N_{n3}^{j+1\,0^*} \ \text{ or } \ u_{3\ell}^i = u_{3\ell}^{i^*},
\end{aligned} \tag{6}$$

$k = \overline{1,K}$, $\ell = \overline{1,L}$, $i = \overline{1,I}$.

Here inferior indexes n and τ are the directions normal and tangential to the edge of the plate, the variables with asterisk indicates boundary value which should be given.

The boundary conditions for the case of simply supported rectangular plate $A \times B$ are given below

at $x_1 = 0$, $x_1 = A$

$$\sum_{i=1}^{I} N_{11}^{i0} = 0, \quad u_2 = 0, \quad u_3 = 0, \quad N_{11}^{ik} + h_i^k \sum_{j=i}^{I-1} N_{11}^{j+1\,0} = 0, \quad u_{2k}^i = 0, \quad u_{3\ell}^i = 0,$$

at $x_2 = 0$, $x_2 = B$

$$u_1 = 0, \quad \sum_{i=1}^{I} N_{22}^{i0} = 0, \quad u_3 = 0, \quad u_{1k}^i = 0, \quad N_{22}^{ik} + h_i^k \sum_{j=i}^{I-1} N_{22}^{j+1\,0} = 0, \quad u_{3\ell}^i = 0, \tag{7}$$

$k = \overline{1,K}$, $\ell = \overline{1,L}$, $i = \overline{1,I}$.

The equations of motion and boundary condition can be expressed in term of the displacement functions using accepted hypotheses (4), the stress–strain relation, the strain-displacement relation and the formulas for stress resultant. The equations of motion (5) in term of the displacement functions may be written as

$$\Omega \overline{U}_{,tt} - \Lambda \overline{U} = \overline{Q}, \tag{8}$$

where $\overline{U}$ is a vector whose components are the sought for functions

$$\overline{U}^T = \left(u_1, u_2, u_3,\ u_{1k}^i, u_{2k}^i, u_{3\ell}^i\right), \qquad i = \overline{1,I},\ k = \overline{1,K},\ \ell = \overline{1,L},$$

Λ, Ω are a square symmetric matrices with the dimensions $((2K+L)I+3) \times ((2K+L)I+3)$; $\overline{Q}$ is a vector whose components depend on external force

$$\overline{Q}^T = (q_1,\ q_2,\ q_3, 0, \ldots, 0).$$

The elements of matrices Ω and Λ are given in Appendices A and B, correspondingly.

The boundary conditions in term of the displacement functions for the case rectangular simply supported plate (7) may be also submitted in the matrix form:

at $x_1 = 0$, $x_1 = A$

$$\left[\Gamma^1_{i,j}\right]\overline{U} = 0, \quad i,j = \overline{1,(2K+L)I+3};$$

at $x_2 = 0$, $x_2 = B$

$$\left[\Gamma^2_{i,j}\right]\overline{U} = 0, \quad i,j = \overline{1,(2K+L)I+3}. \tag{9}$$

The elements of matrices Γ^1 и Γ^2 are given in Appendix C.

The equations of motion (8) and boundary conditions (9) are supplemented by the initial conditions. It is assumed that the initial conditions are equal to zero:

$$\overline{U}|_{t=0} = 0, \quad \overline{U}_{,t}|_{t=0} = 0. \tag{10}$$

Thus, dynamics of multilayer simply supported plate is described by the system of equations of motion (8), boundary (9) and initial (10) conditions.

4. Solution method

The components of the external load $\bar{q}$ as well as displacement functions u_ν, u_3, $u^i_{\nu k}$, $u^i_{3\ell}$ ($\nu = 1,2$, $k = \overline{1,K}$, $\ell = \overline{1,L}$, $i = \overline{1,I}$) are expanded into series by functions $B_{\alpha mn}(x_1,x_2)$ satisfying the boundary conditions

$$\left[u_\alpha, u^i_{\alpha k}, q_\alpha\right] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\Phi_{\alpha mn}(t), \Phi^i_{\alpha kmn}(t), q_{\alpha mn}(t)\right]B_{\alpha mn}(x_1,x_2).$$

In the case of simply supported rectangular plate, functions $B_{\alpha mn}(x_1,x_2)$ have the following form

$$B_{1mn} = \cos\frac{m\pi x_1}{A}\sin\frac{n\pi x_2}{B}, \quad B_{2mn} = \sin\frac{m\pi x_1}{A}\cos\frac{n\pi x_2}{B}, \quad B_{3mn} = \sin\frac{m\pi x_1}{A}\sin\frac{n\pi x_2}{B}.$$

As a result, the problem of non-stationary vibrations of a multilayer plate for each pair of values m and n is reduced to integration of a system of ordinary differential equations with constant coefficient

$$\Omega\overline{\Phi}^{mn}_{,tt} - \Lambda^{mn}\overline{\Phi}^{mn} = \overline{Q}^{mn}, \tag{11}$$

where Λ^{mn} is obtained from Λ by substituting the partial derivatives in the expressions for the matrix elements with the coefficients yielded during differentiation of coordinate functions; $\overline{\Phi}^{mn}$, $\overline{Q}^{mn}$ are vectors

$$(\overline{\Phi}^{mn})^T = \left(\Phi_{1mn}, \Phi_{2mn}, \Phi_{3mn}, \Phi^i_{1kmn}, \Phi^i_{2kmn}, \Phi^i_{3\ell mn}\right), \qquad i = \overline{1,I}, \ k = \overline{1,K}, \ \ell = \overline{1,L},$$

$$(\overline{Q}^{mn})^T = (q_{1mn}, q_{2mn}, q_{3mn}, 0, \ldots, 0)$$

The initial conditions (10) accept a form

$$\overline{\Phi}^{mn}|_{t=0} = 0, \quad \overline{\Phi}^{mn}_{,t}|_{t=0} = 0.$$

The system obtained is integrated by using the modified method of expanding the solution into a Taylor's series.

If the case of static loading is considered, the system (11) is transformed into the following system of algebraic equations

$$\Lambda^{mn}\overline{\Phi}^{mn} = \overline{Q}^{mn}$$

In work for a solution of this system the Gauss method is used.

5. Numerical results

To validate the efficiency of the theory proposed, a test calculation of strain-stressed state of homogeneous and three-layer structures was carried out under static and dynamic loading. Investigated were simply supported structures.

The issues of building operable two-dimensional approximations based on the power series method are discussed. The results of calculations obtained according to the theory proposed are compared with three-dimensional solutions yielded by the method described by Shupikov and Ugrimov (1999).

The strain-stressed state of an infinite homogeneous strip ($h_1 = 0.15$ m, $A = 0.10$ m, $E_1 = 6.0016 \times 10^4$ MPa, $\rho_1 = 2.5 \times 10^3$ kg m^{-3}, $\nu_1 = 0.25$), under static and dynamic loading is considered (Fig. 2). The infinite strip subjected to a loading of the form

$$q_3 = P_0 f(t) \sin\frac{\pi x_1}{A}, \quad q_1 = q_2 = 0,$$

where P_0 is the load intensity, $f(t)$ is a function of time. For static loading $f(t) = 1$, whereas in the case of dynamic loading the function $f(t)$ was chosen to be Heaviside's function:

$$f(t) = H(t) = \begin{cases} 1, & \text{at } t > 0 \\ 0, & \text{at } t \leqslant 0 \end{cases}.$$

The load intensity value is equal to 0.1 MPa in all the cases being investigated.

Table 1 summarizes the results of calculating the static deflection and stresses in terms of the generalized theory at different numbers of retained terms in the series, as well as according to the three-dimensional theory. The results are given for a point located in the middle of the strip outside surfaces.

The results of calculations show that increasing the number of terms retained in expansions (4) increases the generalized theory accuracy and it approaches that of a three-dimensional solution.

Fig. 3 shows the distribution of stresses p_{11}^1 over the plate thickness in its middle section for the case of the static loading. The results of calculations according to the theory described are compared with exact solution given in the paper by Little (1973). The numerical results according to generalized theory {7, 6}

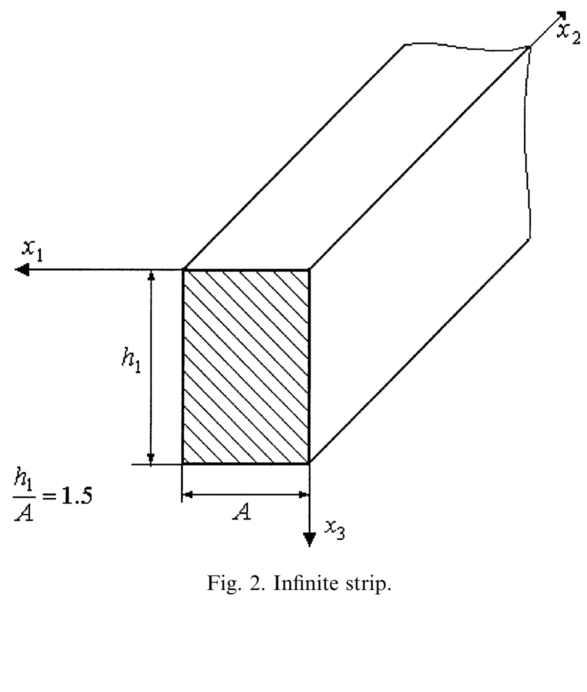

Fig. 2. Infinite strip.

Table 1
Strain-stressed state of the infinite strip under static loading

Generalized theory		Deflections, $u_3^1 \times 10^7$ (m)		In-plane stresses, p_{11}^1 (MPa)		Transverse normal stresses, p_{33}^1 (MPa)	
K	L	$x_3 = 0$	$x_3 = h_1$	$x_3 = 0$	$x_3 = h_1$	$x_3 = 0$	$x_3 = h_1$
1	0	0.33205	0.33205	−0.02702	0.02702	−0.00901	0.00901
3	2	0.96988	0.11167	−0.96378	0.02411	−0.11447	0.01713
5	4	1.00119	0.10431	−0.10479	0.01617	−0.11207	−0.00417
7	6	1.00330	0.10283	−0.10182	0.01684	−0.10114	−0.00065
Three-dimensional theory		1.00360	0.10289	−0.10145	0.01706	−0.1	0

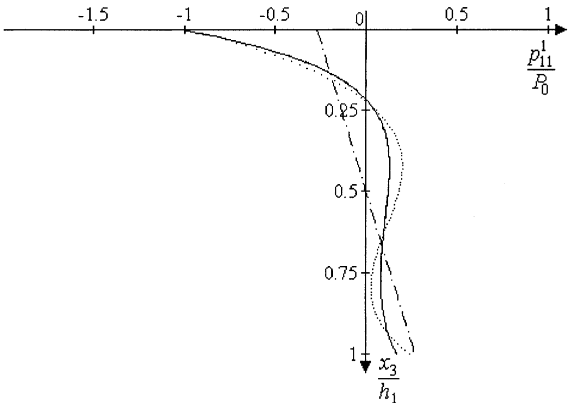

Fig. 3. Distributions of stresses over the thickness in a strip middle section under static loading: solid line, three-dimensional theory; dash-dotted line, theory {1, 0}; dotted line, theory {3, 2}; dashed line, theory {7, 6}.

practically coincide with data obtained on the basis of the three-dimensional theory (distinction does not exceed 1.5%), therefore in the figures they are indistinguishable.

In case of dynamic loading, the similar distribution of stresses over the strip thickness is presented in Fig. 4. This figure shows the stress distributions over the thickness at different time instances corresponding to the travel time of the dilatational wave over 1/2, 1 and $1\frac{1}{2}$ strip thickness respectively. The time of travel of a dilatational wave over the strip thickness is

$$\tau = \frac{h_1}{V} \approx 27.94\ \mu\text{s},$$

where V is the dilatational wave velocity, $V = \sqrt{\frac{\lambda^1 + 2\mu^1}{\rho_1}}$ (Novatsky, 1975).

Due to the fact that two-dimensional theories do not describe the process of wave propagation in the transverse direction, the results obtained in terms of the generalized and three-dimensional theories differ to some extent in their character. Only the three-dimensional solution yields a comprehensive presentation of the wave propagation process over the thickness. However, wave processes in real structures can be observed, as a rule, only during the initial strain stage and over time they decay rapidly. Therefore, the two-dimensional approximation makes it possible to evaluate the strip's strain-stressed state with a sufficient degree of accuracy (Fig. 4).

In the thick strip strain problem being considered the distribution of stresses over the thickness, both for the static and dynamic cases, has an essentially non-linear character, which is poorly approximated by the linear law. Hence, the implementation of theory {1, 0} in this case yields significant errors.

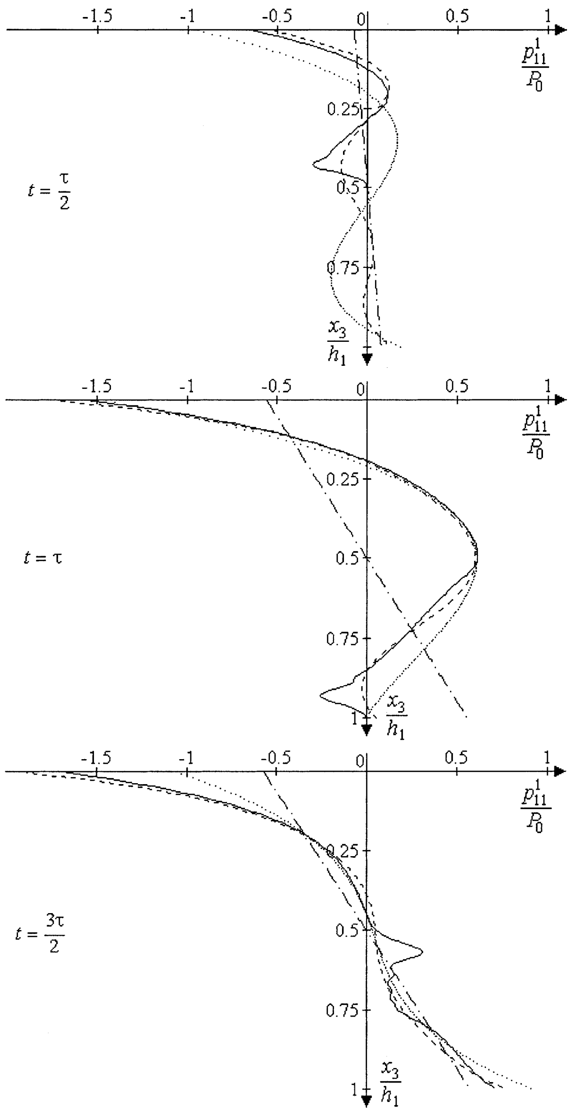

Fig. 4. Distributions of stresses over the thickness in a strip middle section under dynamic loading: solid line, three-dimensional theory; dash-dotted line, theory {1, 0}; dotted line, theory {3, 2}; dashed line, theory {7, 6}.

The three-layer plate is considered ($A = B = 0.3$ m, $h_1 = h_2 = h_3 = 0.01$ m). The first layer of plate is made of organic glass ($E_1 = 5.59 \times 10^3$ MPa, $\nu_1 = 0.38$, $\rho_1 = 1.2 \times 10^3$ kg m^{-3}), the second layer is made of polymeric material ($E_2 = 2.74 \times 10^2$ MPa, $\nu_2 = 0.38$, $\rho_2 = 1.2 \times 10^3$ kg m^{-3}) and third layer is made of silica glass ($E_3 = 6.67 \times 10^4$ MPa, $\nu_3 = 0.22$, $\rho_3 = 2.5 \times 10^3$ kg m^{-3}).

The plate is effected by the load

$$q_3 = \frac{P_0}{2}[H(t) - H(t - t_1)] \sin\frac{\pi t}{t_1} \sin\frac{\pi x_1}{A} \sin\frac{\pi x_2}{B}, \quad q_1 = q_2 = 0,$$

where $P_0 = 0.1$ MPa, $t_1 = 5 \times 10^{-3}$ s.

Tables 2–4 summarize the maximum deflections and stresses in the plate. These values are given for each layer in the points $x_1 = x_2 = 0.15$ m, $x_3 = \delta_{i-1}$ and $x_3 = \delta_i$ $(i = \overline{1,3})$.

The data given in the Tables 2–4 make it possible to estimate the accuracy of two-dimensional approximation when parameters K and L are increased.

As it is shown from Tables 2 and 3, the maximum values of deflections and stresses calculated by the generalized theory with parameters {3, 2}, {5, 4} and {7, 6} practically coincide with each other and are close to the three-dimensional solution. In as much as generalized theory {1, 0} does not allow for pack reduction, the deflections calculated with the help of this theory are constant over the thickness and their maximum value is less than the actual one. In this case theory {1, 0} yields increased values of deflection stresses on the plate's outside surfaces and the maximum difference of 37% is observed in the first layer. Nevertheless, the values of tensile stress calculated by the generalized theory {1, 0} and the three-dimensional one for the load-free backside of the plate are close (see Table 3) and differ by no more than 3%.

Analysis of the results given in Table 4 demonstrates that the usage of the generalized theory {7, 6} makes possible accurate fulfillment of the interlayer contact conditions ($p^i_{33} = p^{i+1}_{33}$ at $x_3 = \delta_i$) and of the boundary conditions on the plate's outside surfaces ($p^1_{33} = -q_3$ at $x_3 = 0$, $p^3_{33} = 0$ at $x_3 = \delta_3$). Generalized theory {3, 2} gives a satisfactory description of the transverse normal stress with an error no more than 12%.

Table 2
Deflections of three-layer plates under impulse loading

Generalized theory		Deflections $u^i_3(A/2, B/2, x_3, t) \times 10^3$ (m) $(x_3 = \delta_{i-1}/x_3 = \delta_i)$		
K	L	1 layer $(i = 1)$	2 layer $(i = 2)$	3 layer $(i = 3)$
1	0	0.12662/0.12662	0.12662/0.12662	0.12662/0.12662
3	2	0.13814/0.13945	0.13945/0.13880	0.13880/0.13873
5	4	0.13814/0.13945	0.13945/0.13880	0.13880/0.13873
7	6	0.13800/0.13931	0.13931/0.13866	0.13866/0.13859
Three-dimensional theory		0.14076/0.14210	0.14210/0.14143	0.14143/0.14136

Table 3
Inplane stresses of three-layer plates under impulse loading

Generalized theory		Stresses $p^i_{11}(A/2, B/2, x_3, t)$ (MPa) $(x_3 = \delta_{i-1}/x_3 = \delta_i)$		
K	L	1 layer $(i = 1)$	2 layer $(i = 2)$	3 layer $(i = 3)$
1	0	−2.46019/−0.21346	−0.01049/−0.04719	−5.64165/7.81009
3	2	−1.76882/−0.42842	−0.06900/−0.05946	−5.47748/7.46842
5	4	−1.76195/−0.42267	−0.06860/−0.05909	−5.47758/7.46459
7	6	−1.75989/−0.42227	−0.06858/−0.05902	−5.47210/7.45665
Three-dimensional theory		−1,79392/−0.43075	−0.06995/−0.06172	−5.58163/7.60581

Table 4
Transverse normal stresses of three-layer plate under impulse loading

Generalized theory		Stresses $p^i_{33}(A/2, B/2, x_3, t)$ (MPa) $(x_3 = \delta_{i-1}/x_3 = \delta_i)$		
K	L	1 layer $(i = 1)$	2 layer $(i = 2)$	3 layer $(i = 3)$
1	0	−1.86972/−0.16223	−0.00797/−0.03642	−2.48235/3.43643
3	2	−0.11144/−0.09126	−0.08273/−0.05085	−0.05182/0.01159
5	4	−0.10032/−0.08189	−0.08207/−0.05023	−0.05018/−0.00001
7	6	−0.10001/−0.08207	−0.08208/−0.05016	−0.05015/0.00000
Three-dimensional theory		−0.10000/−0.08372	−0.08372/−0.05112	−0.05112/0.00000

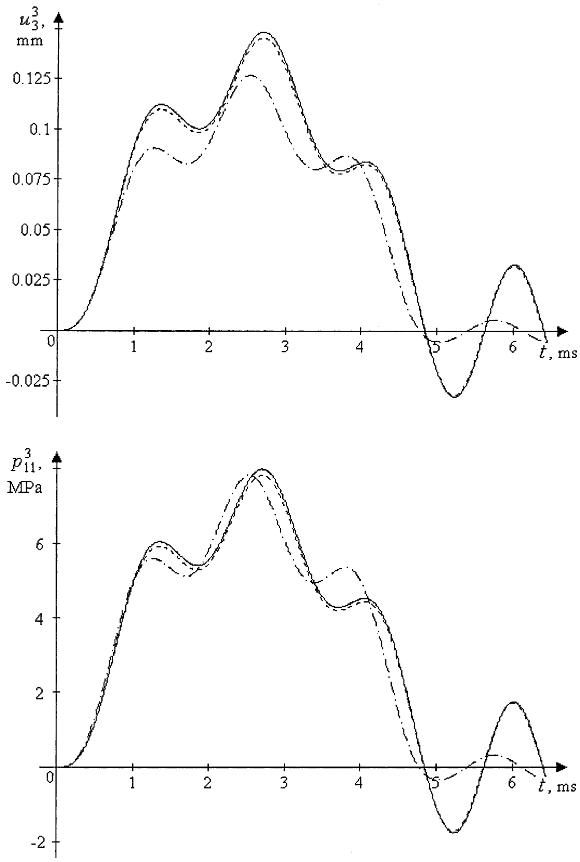

Fig. 5. Deflections and stresses in the plate middle section on the outside surface of the third layer: solid line, three-dimensional theory; dash-dotted line, theory {1, 0}; dashed line, theory {7, 6}.

In Fig. 5 time dependencies of deflections and stresses in the plate middle section on the outside surface of the third layer are presented. The deflection and stress history calculated according to the generalized and three-dimensional theories has the same character. However, if the maximum values of the tensile stress on the outside surface of the third layer practically coincide in all theories, the maximum values of displacements obtained with the help of the generalized theory {1, 0} are significantly less than the actual ones.

Fig. 6 shows variation on time of transverse normal stresses p_{33}^i in the middle on external surfaces for each layer of plate. To the left of the graphs one can see points indicated on the plate composition. These are the points in which the stresses p_{33}^i are calculated. The results obtained on the basis of the generalized theory with parameters {7, 6} and {3, 2} are compared to a three-dimensional solution. On the contact surfaces, the stresses calculated by the generalized theory are given for the i and $i-1$ layers. It is seen that for the time intervals investigated, the results of calculations based on theory {7, 6} and the three-dimensional theory practically coincide and the contact conditions are fulfilled accurately. Though the generalized theory {3, 2} yields a solution close in character to the actual one, the error of approximation of transverse normal stresses on the external surfaces is still significant. In this case it is impossible to fulfill the interlayer contact conditions accurately.

The three-layer plate ($A = B = 0.5$ m) subjected impulse loading is considered. The impulse load have form

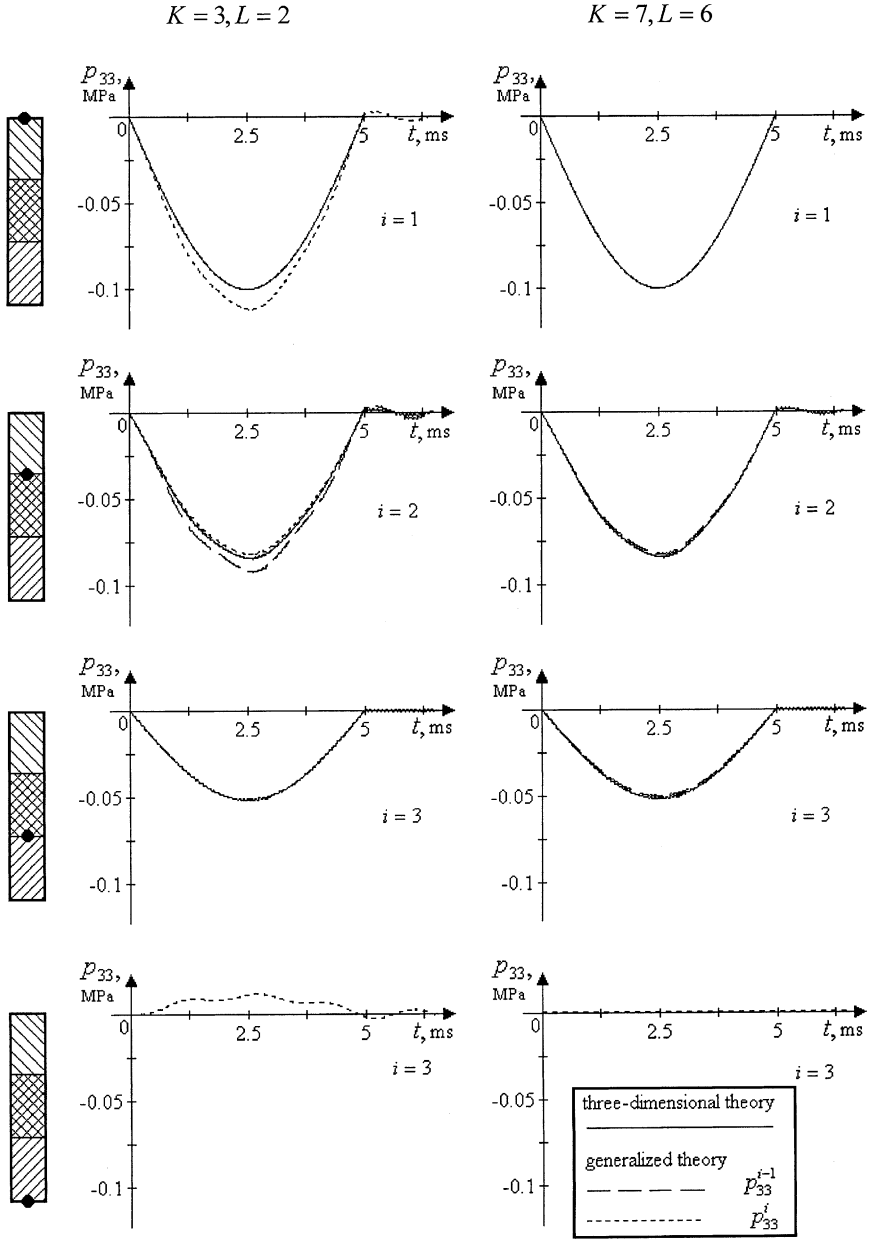

Fig. 6. Transverse normal stresses.

$$q_3 = P_0 H(t) \sin\frac{\pi x_1}{A} \sin\frac{\pi x_2}{B}, \quad q_1 = q_2 = 0,$$

where $P_0 = 0.1$ MPa.

The limits of applicability of two-dimensional theories, depending on the filler susceptibility in a three-layer plate whose external layers are made of silica glass ($E_1 = E_3 = E = 6.67 \times 10^4$ MPa, $\nu_1 = \nu_3 = 0.22$,

$\rho_1 = \rho_3 = 2.5 \times 10^3$ kg m^{-3}), are investigated. The middle layer elasticity modulus E_2 was changed within the following limits: $10^{-6}E \leqslant E_2 \leqslant E$. It is assumed that the other mechanical characteristics (Poisson's ratio, density) of the average layer are equal to the characteristics of silica glass ($\nu_1 = \nu_2 = \nu_3$, $\rho_1 = \rho_2 = \rho_3$). The thickness of layers is $h_1 = h_2 = h_3 = 0.01$ m.

Calculations were carried out with the help of the generalized theory of multilayer plates, the Ambartsumyan's classical theory (1974) (hypotheses of unique non-strained normal for the pack) and three-dimensional theory.

The calculation results are reduced in a Fig. 7. Dependence of maximum stresses in the middle of the plate outside surfaces from a modulus elasticity of an average layer here is shown. Fig. 7 also presents the inplane stress distribution character over the plate thickness for three ratios of E_2/E_1. The distributions are

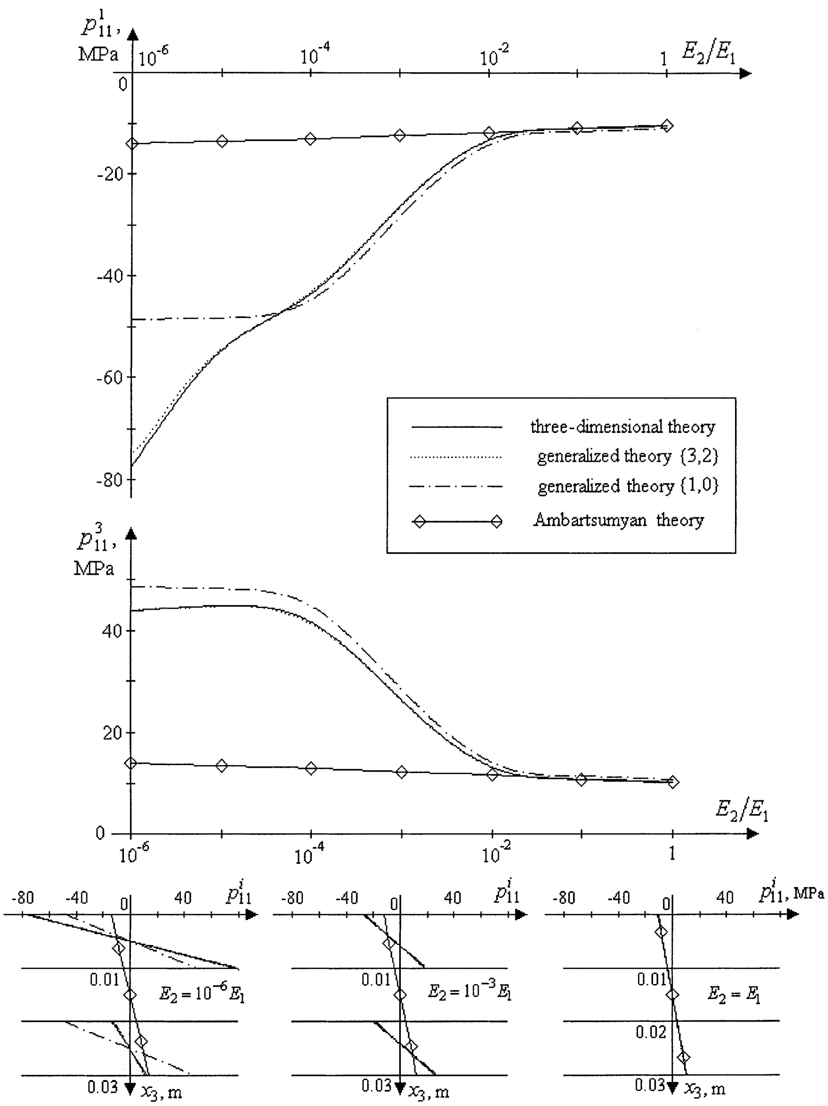

Fig. 7. Influence of a modulus elasticity of an average layer of three-layer plate on maximum values of inplane stresses.

given for time instances when they achieve their maximum values on the outside surface of the first layer. The results obtained by using the generalized theory {7, 6} are not shown in the figure since they practically coincide with the three-dimensional solution (the difference makes up 2%).

It is seen from Fig. 7, that an increase in the middle layer elasticity modulus leads to a stress decrease and at $E_2 = E$ (homogeneous plate) it achieves its minimum value. The stress distributions over the thickness obtained with the help of the generalized and three-dimensional theories are linear at $E_2 = E$. When the elasticity modulus decreases it becomes piecewise-linear, whereas the distribution obtained by the classical theory remains linear. Ambartsumyan's theory closely fits the three-dimensional solution at values of E_2 which satisfy the inequality $E \geqslant E_2 \geqslant 10^{-2}E$; theory {1, 0} yields a close fit at $E \geqslant E_2 \geqslant 10^{-4}E$, whereas the theory with the parameters {7, 6} or {3, 2} may be used for practically all investigated values of E_2.

Fig. 8 shows the specific distribution of transverse shear stresses p_{13}^i over the thickness of the plate being considered for three ratios of E_2/E_1. The stresses are given for the point $x_1 = 0.125$ m, $x_2 = 0.25$ m. It is seen from Fig. 8 that the distribution of the transverse shear stresses over the plate thickness has a

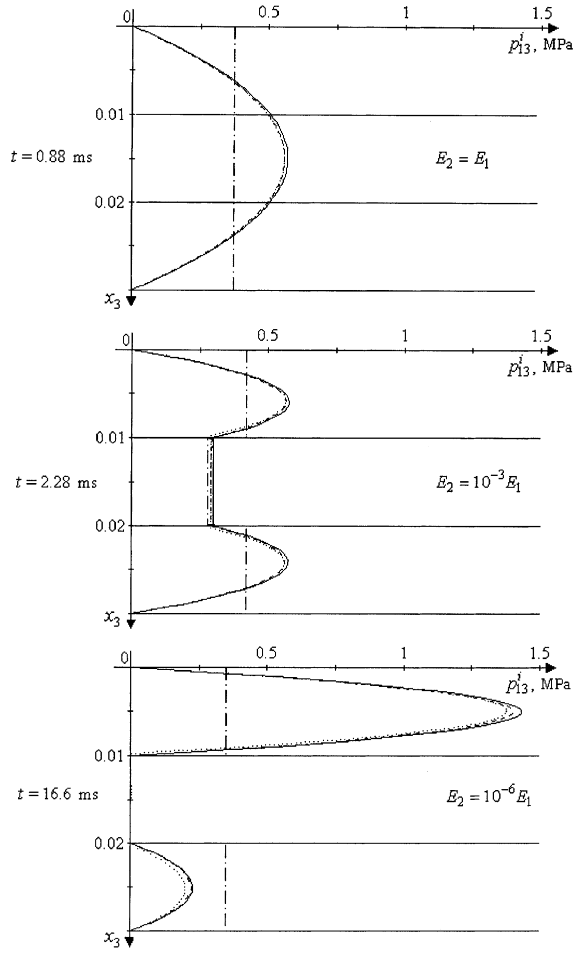

Fig. 8. Variations of transverse shear stress through the thickness of a three-layer plate for different modulus elasticity of an average layer: solid line, three-dimensional theory; dash-dotted line, theory {1, 0}; dotted line, theory {3, 2}; dashed line, theory {7, 6}.

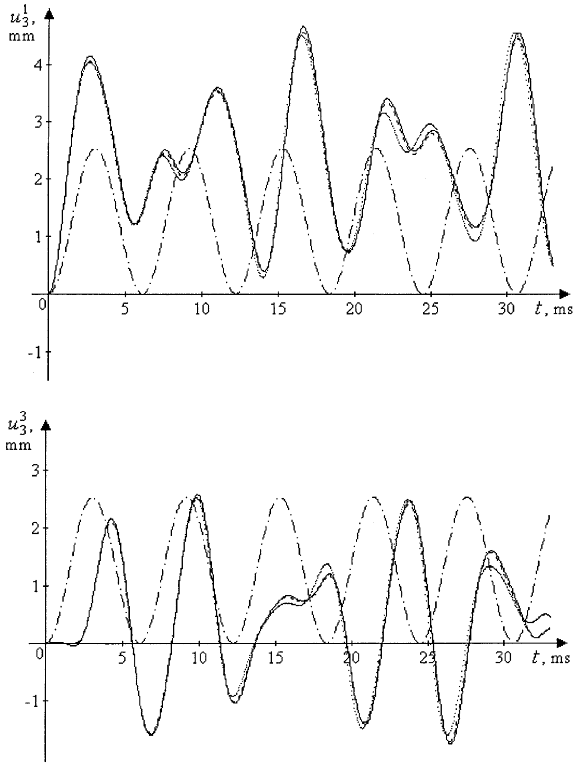

Fig. 9. Deflections of outside surfaces of three-layer plate with a soft filler: solid line, three-dimensional theory; dash-dotted line, theory {1, 0}; dotted line, theory {3, 2}; dashed line, theory {7, 6}.

non-linear character. The generalized theory with parameters {3, 2} and {7, 6} allows obtaining a distribution that is very close to the actual one. The application of theory {1, 0}, which takes into account shear stresses averaged over the layer thickness, makes it impossible to describe accurately the stress distribution character over the thickness, as well as to fulfill (at least approximately) the interlayer contact conditions.

Fig. 9 shows the change of deflections vs. time in the middle of the plate outside surfaces ($x_1 = x_2 = 0.25$ m, $x_3 = 0$ and $x_3 = 0.03$ m) for the same plate in the case of soft filler (at $E_2/E_1 = 10^{-6}$). During strain of the plate being considered there appear significant transverse strains in the soft filler. Therefore, the application of theories, which do not allow for this effect, to similar structures leads to significant errors both in maximum deflection values and in describing the plate deflection change vs. time. It is evident from Fig. 9 that the results obtained according to the generalized theory with parameters {3, 2} and {7, 6}, which take into account the transverse strains, practically coincide with the three-dimensional solution.

6. Conclusions

The work sets forth the generalized two-dimensional theory of multilayer plates based on the power series method, which makes it possible to take into account an arbitrary number of series terms in the expansions of the sought for functions about the transverse coordinate.

The numerical investigations carried out show a close fit of results obtained on the basis of the theory proposed with account of a sufficient number of power series terms and the three-dimensional elasticity theory for a representative set of multilayer plates.

A good solution convergence and the possibility of accurate determination of displacements, and inplane stresses, as well as transverse normal and shear stresses, whose distribution over the thickness in some cases has an essentially non-linear character, have been demonstrated.

This is especially important when investigating the affect of localized loads, for irregularly structured multilayer objects having significantly differing mechanical properties of adjacent layers, as well as when solving the problem of delamination.

The theory proposed has a wide field of application and allows for a valid description of the non-stationary (dynamic) response of multilayer plates having a practically any composition of layers and pack thickness. A limitation of the given theory's field of application is the impossibility to investigate the processes of elastic wave propagation over the plate thickness.

A positive feature is the fact that numerical implementation of the problem of investigating the non-stationary deflected mode of a multilayer plate on the basis of the theory proposed requires fewer resources than when using the three-dimensional elasticity theory.

Appendix A

The elements of the lower triangle of matrix Ω have the following form

$$\Omega_{1,1} = \Omega_{2,2} = -\Omega_{3,3} = \xi_1, \quad \Omega_{3+i+(k-1)I,1} = \Omega_{3+KI+i+(k-1)I,2} = h_i^k S_{ik}, \quad \Omega_{3+2KI+i+(p-1)I,3} = -h_i^p S_{ip},$$

$$\Omega_{3+i+(k-1)I,3+j+(r-1)I} = \Omega_{3+KI+i+(k-1)I,3+KI+j+(r-1)I} = h_i^k h_j^r \begin{cases} S_{ik}, & j < i \\ S_{ik+r}, & j = i \\ S_{jr}, & j > i, \end{cases}$$

$$\Omega_{3+2KI+i+(p-1)I,3+2KI+j+(\ell-1)I} = -h_i^p h_j^\ell \begin{cases} S_{ip}, & j < i \\ S_{ip+\ell}, & j = i, \\ S_{jp}, & j > i \end{cases} \quad i,j = \overline{1,I}, \ \ k,r = \overline{1,K}, \ \ \ell,p = \overline{1,L},$$

where

$$\xi_n = \sum_{i=n}^{I} h_i \rho_i, \quad S_{ik} = \xi_{i+1} + \frac{h_i \rho_i}{k+1}.$$

The remaining matrix elements for lower triangle are equal to zero.

Appendix B

The elements of the lower triangle of matrix Λ have the form

$$\Lambda_{1,1} = C_{11}\frac{\partial^2}{\partial x_1^2} + C_{21}\frac{\partial^2}{\partial x_2^2}, \quad \Lambda_{2,1} = C_{31}\frac{\partial^2}{\partial x_1 \partial x_2}, \quad \Lambda_{2,2} = C_{21}\frac{\partial^2}{\partial x_1^2} + C_{11}\frac{\partial^2}{\partial x_2^2},$$

$$\Lambda_{3,1} = \Lambda_{3,2} = 0, \quad \Lambda_{3,3} = -C_{21}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right), \quad \Lambda_{3+i+(k-1)I,1} = h_i^k\left(D_{1ik}\frac{\partial^2}{\partial x_1^2} + D_{2ik}\frac{\partial^2}{\partial x_2^2}\right),$$

$$\Lambda_{3+i+(k-1)I,2} = \Lambda_{3+K1I+i+(k-1)I,1} = h_i^k D_{3ik} \frac{\partial^2}{\partial x_1 \partial x_2}, \quad \Lambda_{3+i+(k-1)I,3} = -h_i^k \beta_{2i} \frac{\partial}{\partial x_1},$$

$$\Lambda_{3+i+(k-1)I,3+j+(r-1)I} = \eta_{1ikjr} \frac{\partial^2}{\partial x_1^2} + \eta_{2ikjr} \frac{\partial^2}{\partial x_2^2} + \chi_{2ikjr},$$

$$\Lambda_{3+KI+i+(k-1)I,2} = h_i^k \left(D_{2ik} \frac{\partial^2}{\partial x_1^2} + D_{1ik} \frac{\partial^2}{\partial x_2^2} \right), \quad \Lambda_{3+KI+i+(k-1)I,3} = -h_i^k \beta_{2i} \frac{\partial}{\partial x_2},$$

$$\Lambda_{3+KI+i+(k-1)I,3+j+(r-1)I} = \eta_{3ikjr} \frac{\partial^2}{\partial x_1 \partial x_2}, \quad \Lambda_{3+2KI+i+(p-1)I,1} = h_i^p \beta_{4i} \frac{\partial}{\partial x_1},$$

$$\Lambda_{3+KI+i+(k-1)I,3+KI+j+(r-1)I} = \eta_{2ikjr} \frac{\partial^2}{\partial x_1^2} + \eta_{1ikjr} \frac{\partial^2}{\partial x_2^2} + \chi_{2ikjr},$$

$$\Lambda_{3+2KI+i+(p-1)I,2} = h_i^p \beta_{4i} \frac{\partial}{\partial x_2}, \quad \Lambda_{3+2KI+i+(p-1)I,3} = -h_i^p D_{2ip} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right),$$

$$\Lambda_{3+2KI+i+(p-1)I,3+j+(r-1)I} = \zeta_{ipjr} \frac{\partial}{\partial x_1}, \quad \Lambda_{3+2KI+i+(p-1)I,3+KI+j+(r-1)I} = \zeta_{ipjr} \frac{\partial}{\partial x_2},$$

$$\Lambda_{3+2KI+i+(p-1)I,3+2KI+j+(\ell-1)I} = -\eta_{2ipj\ell} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \chi_{1ipj\ell}, \quad i,j = \overline{1,I}, \;\; k,r = \overline{1,K}, \;\; \ell,p = \overline{1,L}.$$

Here

$$C_{\alpha i} = \sum_{j=i}^{I} h_j \beta_{\alpha j}, \quad D_{\alpha is} = C_{\alpha i+1} + \frac{h_i \beta_{\alpha i}}{s+1},$$

$$\beta_{1i} = \frac{E_i(1-v_i)}{(1+v_i)(1-2v_i)}, \quad \beta_{2i} = \frac{E_i}{2(1+v_i)}, \quad \beta_{3i} = \frac{E_i}{2(1+v_i)(1-2v_i)}, \quad \beta_{4i} = \frac{E_i v_i}{(1+v_i)(1-2v_i)},$$

$$\eta_{\alpha ikjr} = h_i^k h_j^r \begin{cases} D_{\alpha ik}, & j < i \\ D_{\alpha ik+r}, & j = i \\ D_{\alpha jr}, & j > i \end{cases}, \quad \chi_{\alpha ikjr} = \begin{cases} 0, & j < i \\ -h_i^{k+r-1} \frac{kr\beta_{\alpha i}}{k+r-1}, & j = i \\ 0, & j > i \end{cases}, \quad \zeta_{ikjr} = h_i^p h_j^r \begin{cases} \beta_{4i}, & j < i \\ \frac{\beta_{4i} p - \beta_{2i} r}{r+p}, & j = i \\ -\beta_{2j}, & j > i \end{cases}.$$

Appendix C

The elements of matrices Γ^1 and Γ^2 have the form

$$\Gamma^1_{1,1} = C_{11} \frac{\partial}{\partial x_1}, \quad \Gamma^1_{1,2} = C_{41} \frac{\partial}{\partial x_2}, \quad \Gamma^1_{1,3+j+(r-1)I} = h_j^r D_{4jr} \frac{\partial}{\partial x_1},$$

$$\Gamma^1_{1,3+KI+j+(r-1)I} = h_j^r D_{4jr} \frac{\partial}{\partial x_2}, \quad \Gamma^1_{1,3+2KI+j+(\ell-1)I} = h_j^\ell \beta_{4j},$$

$$\Gamma^1_{2,2} = \Gamma^1_{3,3} = \Gamma^1_{3+KI+i+(k-1)I,3+KI+i+(k-1)I} = \Gamma^1_{3+2KI+i+(p-1)I,3+2KI+i+(p-1)I} = 1,$$

$$\Gamma^1_{3+i+(k-1)I,1} = h_i^k D_{1ik}\frac{\partial}{\partial x_1}, \quad \Gamma^1_{3+i+(k-1)I,2} = h_i^k D_{4ik}\frac{\partial}{\partial x_2},$$

$$\Gamma^1_{3+i+(k-1)I,3+j+(r-1)I} = \eta_{1ikjr}\frac{\partial}{\partial x_1}, \quad \Gamma^1_{3+i+(k-1)I,3+KI+j+(r-1)I} = \eta_{4ikjr}\frac{\partial}{\partial x_2},$$

$$\Gamma^1_{3+i+(k-1)I,3+2KI+j+(\ell-1)I} = h_i^k h_j^\ell \beta_{4j}\begin{cases} 0, & j < i \\ \frac{\ell}{k+\ell}, & j = i \\ 1, & j > i \end{cases},$$

$$\Gamma^2_{1,1} = \Gamma^2_{3,3} = \Gamma^2_{3+i+(k-1)I,3+i+(k-1)I} = \Gamma^2_{3+2KI+i+(p-1)I,3+2KI+i+(p-1)I} = 1,$$

$$\Gamma^2_{2,1} = C_{41}\frac{\partial}{\partial x_1}, \quad \Gamma^2_{2,2} = C_{11}\frac{\partial}{\partial x_2}, \quad \Gamma^2_{2,3+j+(r-1)I} = h_j^r D_{4jr}\frac{\partial}{\partial x_1},$$

$$\Gamma^2_{2,3+KI+j+(r-1)I} = h_j^r D_{1jr}\frac{\partial}{\partial x_2}, \quad \Gamma^2_{2,3+2KI+j+(\ell-1)I} = h_j^\ell \beta_{4j},$$

$$\Gamma^2_{3+KI+i+(k-1)I,1} = h_i^k D_{4ik}\frac{\partial}{\partial x_1}, \quad \Gamma^2_{3+KI+i+(k-1)I,2} = h_i^k D_{1ik}\frac{\partial}{\partial x_2},$$

$$\Gamma^2_{3+KI+i+(k-1)I,3+j+(r-1)I} = \eta_{4ikjr}\frac{\partial}{\partial x_1}, \quad \Gamma^2_{3+KI+i+(k-1)I,3+KI+j+(r-1)I} = \eta_{1ikjr}\frac{\partial}{\partial x_2},$$

$$\Gamma^2_{3+KI+i+(k-1)I,3+2KI+j+(\ell-1)I} = h_i^k h_j^\ell \beta_{4j}\begin{cases} 0, & j < i \\ \frac{\ell}{k+\ell}, & j = i, \quad i = \overline{1,I}, \quad k = \overline{1,K}, \quad \ell = \overline{1,L}. \\ 1, & j > i \end{cases}$$

The remaining elements of matrices Γ^1 and Γ^2 are equal to zero.

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